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NOTE ON THE EQUILATERAL HYPERBOLA

BY JOHN A. VAN GROOS

1. THE equilateral hyperbola $x^2 - y^2 = 1$ can be represented by the equations:

$$x = \cosh t, \qquad y = \sinh t,$$

where the right branch of the hyperbola corresponds to the values of t from $-\infty$ to $+\infty$, and the left branch to the values from $-\infty + \pi i$ to $+\infty + \pi i$.

When $t = 0$ we get the point $(1, 0)$; when $t = 0 + \pi i$, the point $(-1, 0)$. The points $-t$ and $-t + \pi i$ are symmetrical to the point t with respect to the axes of X and Y respectively; the points t and $t \pm 2n\pi i$ are of course identical. If the pure imaginary part of t is not a multiple of π, the corresponding point (x, y) will be imaginary.

For real values of the parameter, t equals twice the area of a sector whose sides are the major axis and the radius vector to the point t.

2. *If the three vertices t_1, t_2, t_3, of a triangle lie on the curve, then its orthocentre (point of intersection of the altitudes) will also lie on the curve.*

This well known theorem is due to Brianchon and Poncelet,* and is easily proved as follows:

Let the three vertices be $t_1 = (x_1, y_1)$, $t_2 = (x_2, y_2)$, and $t_3 = (x_3, y_3)$; then the equation of the line $t_1 t_2$ is

$$\frac{y - y_1}{x - x_1} = \frac{y_1 - y_2}{x_1 - x_2} = \frac{\sinh t_1 - \sinh t_2}{\cosh t_1 - \cosh t_2} = \coth \tfrac{1}{2}(t_1 + t_2),$$

and the equation of the perpendicular to this line through t_3 is

$$(1) \qquad \frac{y - y_3}{x - x_3} = -\tanh \tfrac{1}{2}(t_1 + t_2).$$

Let $t_0 = (x_0, y_0)$ be the point where this perpendicular cuts the hyperbola. Then we have

$$(2) \qquad \coth \tfrac{1}{2}(t_0 + t_3) = -\tanh \tfrac{1}{2}(t_1 + t_2),$$

$$(3) \quad \text{whence} \qquad t_0 \equiv -(t_1 + t_2 + t_3) + \pi i \qquad (\text{mod } 2\pi i).$$

Gergonne's Annales, vol. 11, pp. 205-220, Jan. 1, 1821.

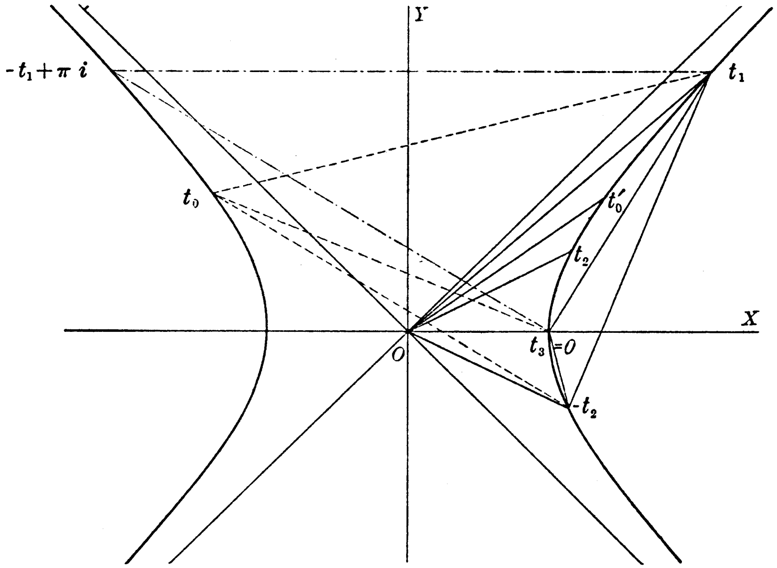
If we take a second altitude of the triangle, we find, in the same way, that this line cuts the hyperbola in the same point t_0.

3. From the relation (3) we deduce at once the theorem:

If 3^n points on an equilateral hyperbola be grouped in 3^{n-1} triangles in any way, they determine 3^{n-1} orthocentres. These orthocentres grouped in any way determine 3^{n-2} new orthocentres, and so on. The last three points determine a final point on the hyperbola. No matter how the points are grouped into triangles, the final point is always the same, its parameter being

$$t_0 \equiv -\sum_{\kappa=1}^{3^n} t_\kappa + \pi i \qquad (\text{mod } 2\pi i). \tag{4}$$

4. The relation (3) enables us also to add geometrically any two sectors of an equilateral hyperbola.



First, to reduce any sector $t_1 O t_2$ (see figure) to an equivalent sector having the major axis (from O to $t_3 = 0$) as a side, find the point $-t_2$ symmetric to t_2 with respect to the X-axis, construct the orthocentre t_0 of the points $t_1, -t_2, t_3$, and find its symmetric point t_0' with respect to the Y-axis; then the sector under t_0' will be the sector required. For, by (3), $t_0 = -(t_1 - t_2 + 0) + \pi i$, and therefore $t_0' = -t_0 + \pi i = t_1 - t_2$.

Secondly, to add two sectors which have the major axis as a common side, as the sectors under t_1 and t_2 in the figure, construct the orthocentre of the points t_1, t_2, t_3, and its symmetric point with respect to the axis of Y; the sector under this last point will be equivalent to the sum of the sectors under t_1 and t_2.

5. If we take two vertices, t_1 and $t_2 = -t_1 + \pi i$, of our inscribed triangle on a line parallel to the X-axis, and let the third vertex fall at a vertex, $t_3 = 0$, of the hyperbola, we shall get for the orthocentre

$$t_0 = -(t_1 - t_1 + \pi i + t_3) + \pi i = -t_3; \tag{5}$$

that is, the orthocentre will fall at $t_3 = 0$, and the triangle will be right angled. *Hence, in an equilateral hyperbola any chord parallel to the principal axis subtends a right angle at either vertex of the curve.*[*]

If now t_3 moves upward to t_1, the orthocentre t_0 will move downward, always keeping symmetric to t_3 with respect to the X-axis, by (5). Hence in the limit, when t_3 reaches t_1, and $t_3\,t_1$ becomes a tangent to the curve at t_1, the point t_0 will be symmetrical to t_1 with respect to the major axis.

Hence, to draw a tangent at any point t_1, *find its symmetric points on the hyperbola with respect to the* x *and* y *axes and join the points so found. Then the perpendicular from* t_1 *upon this joining line will be the required tangent.*

6. If we take four points on the curve so as to form a parallelogram, we can construct four inscribed triangles from these four points. The four orthocentres of these triangles will again form a parallelogram.

If we take four points on the hyperbola so as to form an isosceles trapezoid, the orthocentres formed from the four inscribed triangles will again form an isosceles trapezoid. Taking these last four points and repeating the process, we get the four original points; that is, one may say that an isosceles trapezoid inscribed in an equilateral hyperbola has a periodicity two.

YALE UNIVERSITY, JANUARY, 1903.

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[*] Cf. problem 501 in C. Taylor's *Geometry of Conics* (1881), p. 184: "In any right angled triangle inscribed in an equilateral hyperbola the perpendicular upon the hypothenuse is the tangent at the right angle."